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THE DISCRETE BID FIRST AUCTION

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In a first auction in wich buyers can bid only multiples of an increment and have uniformly distributed values, the expected price is less than the continuous bid expected price.

I discuss the first price auction, within the independent private values model [Milgrom and Weber (1982)], in which the seller auctions an object to N buyers. Each buyer's value of the object is distributed independently over $[\underline{v}, \bar{v}]$ with cumulative distribution function $F(v)$. Each buyer, knowing only her own value of the object, chooses a b_i from the set $\{b_1, b_2, \dots, b_M\}$ (I assume that $b_1 = \underline{v}$ and $b_M < \bar{v}$) and the highest bidder receives the good at a price equal to his bid. If two or more buyers bid the same highest b_i, one of them, chosen randomly and fairly, receives the good at a price equal to b_i. When buyer values are uniformly distributed and bid possibilities are multiples of an increment, I show that a symmetric Nash equilibrium exists uniquely and converges to the equilibrium of the continuous bid auction as the bid increment goes to zero; however, the resulting discrete bid expected price is always less than the continuous bid expected price, and thus the seller has an incentive to make bid increments small.

A buyer's strategy is a function from $[\underline{v}, \bar{v}]$ to $\{b_1, b_2, \dots, b_M\}$ returning the buyer's optimal b_i, given her value of the object v. Strategies we consider are of the (sufficiently general) form:

$$b(v) = \begin{cases} b_i, & \text{if} \quad v \in [s_{i-1}, s_i), \quad 1 \le i \le r, \\ b_r, & \text{if} \quad v = s_r, \end{cases}$$

where $\underline{v} = s_0 < s_1 < s_2 < \cdots < s_r = \bar{v}$, and r is an integer such that $1 \le r \le M$. Let's calculate the expected gain, $EG(v, b_i)$, for a buyer whose value is v and bids b_i, when every other buyer employs the strategy $b(v)$. If $r + 1 \le i \le M$, $EG(v, b_i) = v - b_i$. Otherwise, $EG(v, b_i) = \sum_{t=1}^{N} (1/t)(v - b_i) \Pr(b_i$ is the highest bid and t buyers bid $b_i)$, and so

$$EG(v, b_i) = \sum_{t=1}^{N} (1/t)(v - b_i)\binom{N-1}{t-1}[F(s_i) - F(s_{i-1})]^{t-1}[F(s_{i-1})]^{N-t}$$

$$= [(v - b_i)/(N(F(s_i) - F(s_{i-1})))] \sum_{t=1}^{N} \binom{N}{t}[F(s_i) - F(s_{i-1})]^{t}[F(s_{i-1})]^{N-t}$$

$$= [(v - b_i)/N]\left[(F(s_i))^N - (F(s_{i-1}))^N\right]/[F(s_i) - F(s_{i-1})].$$

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If we let $\Delta_i = [1/N][(F(s_i))^N - (F(s_{i-1}))^N]/[F(s_i) - F(s_{i-1})]$, then $EG(v, b_i) = (v - b_i)\Delta_i$.

The following four conditions might not be independent, but it is fairly straightforward to prove that $b(v)$ is a symmetric (pure strategy) Nash equilibrium strategy if and only if $s_0, s_1, \dots, s_r$ satisfy them. The first condition is that when a buyer's value falls exactly on a s_i, she obtains the same expected gain from bidding b_i and b_{i+1}. So $EG(s_i, b_i) = EG(s_i, b_{i+1})$, or

Condition 1. $(s_i - b_i)\Delta_i = (s_i - b_{i+1})\Delta_{i+1}$, for $1 \le i \le r-1$.

We also need $EG(\bar{v}, b_r) \ge EG(\bar{v}, b_{r+1})$, or

Condition 2. $(\bar{v} - b_r)\Delta_r \ge \bar{v} - b_{r+1}$.

Let's call our earlier assumption

Condition 3. $\underline{v} = s_0$, $\bar{v} = s_r$, and $s_{i-1} < s_i$ for $1 \le i \le r$.

Finally, to ensure that bidders never bid above their value, we need

Condition 4. $s_{i-1} \ge b_i$, for $1 \le i \le r$.

Let values be distributed uniformly: $[\underline{v}, \bar{v}] = [0, 1]$ and $F(v) = v$. Most often, bid possibilities are multiples of a small increment, like a dollar. So let $b_i = (i-1)/M$. As M gets large, the increment goes to zero and the 'grid' of bid possibilities gets finer.

Proposition 1. When $[\underline{v}, \bar{v}] = [0, 1]$, $F(v) = v$, and $b_i = (i-1)/M$, unique r and unique $s_0, s_1, \dots, s_r$ exist which satisfy the four conditions. Also the following inequalities hold for $1 \le i \le r$: (i) $(s_{i-1} + s_i)/2 > Nb_i/(N-1)$; (ii) $s_i \le Nb_{i+1}/(N-1)$; and (iii) $s_i > Nb_i/(N-1)$.

Proof. First we show that Condition 1 is equivalent to Condition 1 (uniform). For $1 \le j \le r$, $(s_j - b_j)\Delta_j = s_j^N/N$. Say $s_0, s_1, \dots, s_r$ satisfy Condition 1 (and hence $s_i(\Delta_i - \Delta_{i+1}) = b_i\Delta_i - b_{i+1}\Delta_{i+1}$ for $1 \le i \le r-1$). For $j = 1$, Condition 1 (uniform) is satisfied trivially. For $2 \le j \le r$, $s_j^N/N =$

$$\sum_{i=1}^{j} (s_i^N - s_{i-1}^N)/N = \sum_{i=1}^{j} (s_i - s_{i-1})\Delta_i = -s_0\Delta_1 + s_j\Delta_j + \sum_{i=1}^{j-1} s_i(\Delta_i - \Delta_{i+1})$$

$$= s_j\Delta_j + \sum_{i=1}^{j-1} b_i\Delta_i - b_{i+1}\Delta_{i+1} = s_j\Delta_j + b_1\Delta_1 - b_j\Delta_j = (s_j - b_j)\Delta_j.$$

Now say that $s_0, s_1, \dots, s_r$ satisfy Condition 1 (uniform). Then for $1 \le j \le r-1$, $(1/N)(s_{j+1}^N - s_j^N) = s_{j+1}\Delta_{j+1} - b_{j+1}\Delta_{j+1} - s_j\Delta_j + b_j\Delta_j$. But from the definition of Δ_j, $(1/N)(s_{j+1}^N - s_j^N) = (s_{j+1} - s_j)\Delta_{j+1}$. Thus $(s_j - b_j)\Delta_j = (s_j - b_{j+1})\Delta_{j+1}$.

Choose r such that $b_r = \max\{b_i\colon\ b_i < (N-1)/N\}$. Such an r exists, since $b_1 = 0 < (N-1)/N$. Since $b_{r+1} \ge (N-1)/N$, $EG(1, b_r) = 1/N \ge 1 - b_{r+1} = EG(1, b_{r+1})$, and so Condition 2 holds (it is also easy to show that (*ii*) and (*iii*) are satisfied for $i = r$). To show that this is the only possible r, first say $b_r > \max\{b_i\colon\ b_i < (N-1)/N\}$. So $b_r \ge (N-1)/N$. We know that $N\Delta_r = 1/(1 - b_r)$ from Condition 1 (uniform) and that $s_{r-1} < s_r = 1$ by Condition 3. But

$$N\Delta_r = (1 - s_{r-1}^N)/(1 - s_{r-1}) = \sum_{j=0}^{N-1} s_{r-1}^j < \sum_{j=0}^{N-1} (1)^j = N.$$

So $1 < N(1 - b_r)$, and hence $b_r < (N-1)/N$, a contradiction. Next assume $b_r < \max\{b_i\colon\ b_i < (N-$

$1)/N\}$. Then $b_{r+1} < (N-1)/N$. But then, $EG(1, b_{r+1}) = 1 - b_{r+1} > 1/N = EG(1, b_r)$, and Condition 2 is violated.

If $r = 1$, the conditions and (*i*)–(*iii*) hold trivially, and we are done. So assume $r \geq 2$, and let $2 \leq i \leq r$. We show that given a s_i which satisfies (*ii*) and (*iii*), a s_{i-1} uniquely exists which satisfies Condition 1 (uniform); this s_{i-1} also satisfies Conditions 3 and 4 and (*i*)–(*iii*). To do this we use the following lemma (the proof is not very interesting and is available upon request):

Lemma 1. Choose γ *such that* $0 < \gamma < 1$ *and let N be an integer greater than* 1. *Then a unique x exists in* (0, 1) *which solves* $(1 - x^N)/(1 - x) = N/(1 + (N-1)\gamma)$, *and: (1)* $x > 1 - 2\gamma$; *(2)* $x < 1 - \gamma$; *and (3)* $x > (N-1)(1-y)/N$.

Let $\gamma = 1 - [N/(N-1)][b_i/s_i]$. Since s_i satisfies (*iii*), $0 < \gamma < 1$, and so we can use Lemma 1 to conclude that there uniquely exists a $0 < (s_{i-1}/s_i) < 1$ which solves $[1 - (s_{i-1}/s_i)^N]/[1 - (s_{i-1}/s_i)] = N/(1 + (N-1)\gamma) = 1/[1 - (b_i/s_i)]$, which is equivalent to Condition 1 (uniform). Since $s_{i-1}/s_i < 1$, Condition 3 holds. (1), (2), and (3) in Lemma 1 imply (*i*), (*ii*), and Condition 4 respectively. From (1), $s_{i-1} > -s_i + 2Nb_i/(N-1)$. Since s_i satisfies (*ii*), $s_i \leq Nb_{i+1}/(N-1)$, and thus $s_{i-1} > [N/(N-1)](2b_i - b_{i+1}) = Nb_{i-1}/(N-1)$, and thus (*iii*) holds. We finish by checking that $s_0 = 0$ satisfies the conditions and (i). Q.E.D.

Let $bid_{N,M}(v)$ be the symmetric Nash strategy defined by the s_i's and let $Ep_{N,M}$ be the resulting expected price. Vickrey (1961) showed that if buyers can make bids in the continuous interval [0, 1], the symmetric Nash equilibrium strategy is $bid(v) = (N-1)v/N$ and the expected price is $(N-1)/(N+1)$.

Corollary 1. As $M \to \infty$, $bid_{N,M}(v)$ *converges uniformly to* $bid(v) = (N-1)v/N$.

Proof. If $s_{i-1} \leq v < s_i$, where $2 \leq i \leq r$, $bid_{N,M}(v) = b_i$. From (*iii*), $Nb_{i-1}/(N-1) < s_{i-1}$, and from (*ii*), $s_i \leq Nb_{i+1}/(N-1)$. So $b_{i-1} < (N-1)v/N < b_{i+1}$. If $s_0 \leq v < s_1$, $bid_{N,M}(v) = b_1$, and $b_1 \leq (N-1)v/N < b_2$ since $s_0 = b_1 = 0$ and (*ii*). If $v = s_r$, $bid_{N,M}(v) = b_r$ and $b_{r-1} < (N-1)v/N \leq b_{r+1}$. So for $0 \leq v \leq 1$, $|(N-1)v/N - bid_{N,M}(v)| \leq 1/M \to 0$ as $M \to \infty$. Q.E.D.

Corollary 2. For all $M \geq 2$ *and* $N \geq 2$, $(N-1)/(N+1) > Ep_{N,M}$.

Proof. For $1 \leq i \leq r$, since $s_i > s_{i-1}$, $s_i[(N-1)/(N+1)] - s_{i-1} > s_{i-1}[(N-1)/(N+1)] - s_i$.

$$s_i^N\left[s_i[(N-1)/(N+1)] - s_{i-1}\right] > s_{i-1}^N\left[s_{i-1}[(N-1)/(N+1)] - s_i\right].$$

$$[(N-1)/(N+1)]\left[s_i^{N+1} - s_{i-1}^{N+1}\right] > s_i^N s_{i-1} - s_i s_{i-1}^N.$$

$$[2N/(N+1)]\left[s_i^{N+1} - s_{i-1}^{N+1}\right] > s_i^N s_{i-1} - s_i s_{i-1}^N + s_i^{N+1} - s_{i-1}^{N+1}.$$

$$[(N-1)/(N+1)]\left[s_i^{N+1} - s_{i-1}^{N+1}\right] > [(N-1)/N]\left[(s_{i-1} + s_i)/2\right]\left[s_i^N - s_{i-1}^N\right].$$

Since, from (*i*), $(s_{i-1} + s_i)/2 > Nb_i/(N-1)$, we can add up inequalities to get

$$[(N-1)/(N+1)] \sum_{i=1}^{r} s_i^{N+1} - s_{i-1}^{N+1} > \sum_{i=1}^{r} b_i\left[s_i^N - s_{i-1}^N\right].$$

So $(N-1)/(N+1) > Ep_{N,M}$. Q.E.D.

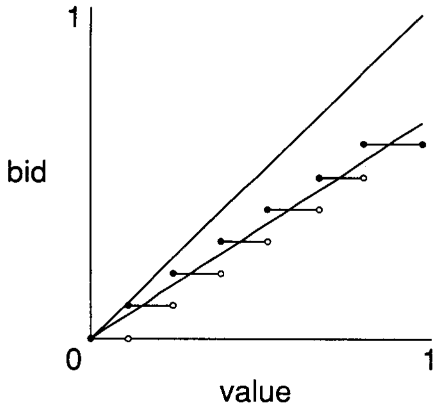

Fig. 1.

Using a computer, we make the following estimates: $Ep_{3,2} = 0.382$, $Ep_{3,3} = 0.317$, $Ep_{3,8} = 0.483$, $Ep_{3,9} = 0.476$, and $Ep_{3,10} = 0.488$, which compare with the continuous bid expected price of 0.500. Note that $Ep_{N,M}$ does not increase monotonically in M – making the set of bid possibilities finer doesn't necessarily increase expected revenue. In fig. 1, $bid_{3,10}(v)$ and $bid(v) = 2v/3$ are graphed.

Our theory is not developed to the point of being directly applicable to experimental auctions (one reason being that buyers in experimental situations tend to be risk averse – for a discussion of discreteness see Cox, Smith and Walker (1986, pp. 15–16). On a more speculative level, we can see how sellers have an incentive to establish small exchange units (money, perhaps). One conclusion at least is that continuous models of discrete institutions can yield approximations which are good but systematically biased.

References

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